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LETTER TO THE EDITOR

Chaos in gauge theories possessing vortices and monopole solutions

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Abstract. We have looked for the signature of chaos in the Abelian Higgs model and SO(3) Georgi-Glashow model which possess vortices and monopole solutions respectively. On applying Painlevé analysis we find that most of the type-I region of superconductivity in the Abelian Higgs model and $\lambda > 2g^2$ region in the Georgi-Glashow model is non-integrable (here λ is the Higgs coupling while g is the gauge coupling constant). Further using the Toda-Brumer criterion we find that the critical energy for the onset of chaos is $E_c = \frac{11}{108} c_2^2/c_4$ and $E_c = m^4/54\lambda$ in the Abelian Higgs model and Georgi-Glashow model respectively.

It is well known that the sine-Gordon and KdV like non-linear systems have infinite conserved quantities and hence are completely integrable systems (Lamb 1980). On the other hand, another non-linear system, i.e. pure Yang-Mills theory, has been shown to be chaotic under suitable assumptions. In fact it has been shown to be a Kolmogorov K-system with strong statistical properties (Baseyan *et al* 1979, Matinyan *et al* 1980, Savvidy 1984). In a way solitons and chaos are paradigms for opposite extremes of non-linear behaviour. Can they coexist together? For example is there chaos in Yang-Mills theory (more generally in gauge theories) possessing topologically non-trivial finite energy solutions? This is the question we would like to raise and partially answer in this letter. In particular, we show that under suitable assumptions the Abelian Higgs model and SO(3) Georgi-Glashow model, which admit vortex (Nielsen and Olesen 1973) and monopole (t'Hooft 1974, Polyakov 1974) solutions, respectively, exhibit chaos. In both cases we confine ourselves to the region where the gauge and the Higgs fields are homogenous in space and reduce the problem to a non-linear mechanical system. By applying Painlevé analysis (Steeb and Louw 1986 and references therein) we find the following.

(i) The Abelian Higgs model is algebraically non-integrable in the case $\lambda^2(\equiv 8c_4/e^2) < \frac{7}{8}$, i.e. in most of the type-I region of superconductivity.

(ii) The SO(3) Georgi-Glashow model is algebraically non-integrable in $\lambda > 2g^2$. In view of the non-integrability we apply the Toda-Brumer criterion (Toda 1974, Brumer and Duff 1976) and find that the critical energy for the onset of chaos is $E_c = \frac{11}{108} c_2^2/c_4$ and $E_c = m^4/54\lambda$ in the above-mentioned two models.

The Lagrangian density for the Abelian Higgs model is given by

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}(D_\mu \phi)^*(D^\mu \phi) + c_2|\phi|^2 - c_4|\phi|^4 \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) and $D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$. The field equations are

$$D_\mu D^\mu \phi = -2c_2\phi + 4c_4\phi^2\phi^* \tag{2a}$$

$$\partial^\nu F_{\mu\nu} = \tfrac{1}{2} ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2 A_\mu \phi^*\phi. \tag{2b}$$

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As has been shown by Nielsen and Olesen (1973), this model admits (static) vortex solutions of vorticity n ($n=\pm1, \pm2, \ldots$). It is also known that the vortex-vortex interaction is attractive in the type-I region of superconductivity (i.e. $\lambda^2 \equiv 8c_4/e^2 < 1$) while it is repulsive in the type-II region ($\lambda^2 > 1$). We shall consider this model in the regime where the space variations are negligible compared to their time variation. We make the ansatz

$$A_0(x, t) = 0 \qquad A_1(x, t) = A_2(x, t) = h(t)/\sqrt{2} \tag{3a}$$

$$\phi(x, t) = \exp[i\omega(x+y)]q_2(t). \tag{3b}$$

Using the ansatz in field equations ($2a$) and ($2b$) we obtain

$$\ddot{h}(t) = -\sqrt{2}e\omega q_2^2 - e^2 h q_2^2 \tag{4a}$$

$$\ddot{q}_2(t) = 2(c_2 - \omega^2)q_2 - 2\sqrt{2}e\omega h q_2 - e^2 h^2 q_2 - 4c_4 q_2^3. \tag{4b}$$

The equations take a simplified form if we make the substitution $h(t) = q_1(t) - \sqrt{2}\omega/e$ in equations ($4a$) and ($4b$). We get

$$\ddot{q}_1(t) = -e^2 q_1 q_2^2 \tag{5a}$$

$$\ddot{q}_2(t) = 2c_2 q_2 - 4c_4 q_2^3 - e^2 q_1^2 q_2. \tag{5b}$$

These equations of motion follow from the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + e^2 q_1^2 q_2^2 + c_4(q_2^2 - c_2/2c_4)^2. \tag{6}$$

It is worthwhile pointing out that in the Yang–Mills case too the Hamiltonian has the non-central piece $q_1^2 q_2^2$ which is responsible for chaotic behaviour. Hence we expect that the H as given by equation (6) will also show chaotic behaviour for some range of parameter values.

Painlevé test. As a first step in the study of chaotic behaviour, let us check if the system as given by equation (6) is integrable or not. To that end one must perform the singular-point analysis and determine the resonances (Steeb and Louw 1986 and references therein). Recent work of Yoshida (1983) tells us that the necessary condition for the system to be algebraically integrable is that every resonance should be a rational number.

The Hamiltonian equations which follow from (6) are

$$\dot{q}_1 = p_1 \qquad \dot{q}_2 = p_2 \tag{7a}$$

$$\dot{p}_1 = -e^2 q_1 q_2^2 \qquad \dot{p}_2 = -e^2 q_1^2 q_2 - 4c_4 q_2^3 + 2c_2 q_2. \tag{7b}$$

Now we have to find the dominant behaviour of q_i and p_i in the complex time (τ) domain by retaining only the leading terms (in our case it amounts to neglecting the term $2c_2 q_2$). On substituting

$$q_i = c_i(\tau - \tau_0)^{-m_i} \tag{8a}$$

$$p_i = d_i(\tau - \tau_0)^{-n_i} \tag{8b}$$

in ($7a$) and ($7b$) we find ($i = 1, 2$)

$$n_i = m_i + 1 \qquad d_i = -c_i \tag{9a}$$

$$e^2 c_1^2 = -2 + 8c_4 e^2 \qquad e^2 c_2^2 = -2. \tag{9b}$$

The resonances (Kowalewski exponents) can now be obtained by substituting

$$q_i = c_i(\tau-\tau_0)^{-1} + b_i(\tau-\tau_0)^{-1+\gamma} \tag{10a}$$

$$p_i = -c_i(\tau-\tau_0)^{-2} + d_i(\tau-\tau_0)^{-1+\gamma} \tag{10b}$$

into (7*a*) and (7*b*). To leading order in b_i and d_i this yields

$$Q(\gamma)\begin{pmatrix} b_i \\ d_i \end{pmatrix} = 0 \tag{11}$$

where $Q(\gamma)$ is a 4×4 matrix whose elements depend on γ. The roots of det Q are the resonances. In our case we find that the resonances are

$$-1, 4, \tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{64c_4/e^2-7}. \tag{12}$$

Two of the resonances are complex if $\lambda^2(\equiv 8c_4/e^2)<\frac{7}{8}$ so that for $\lambda^2<\frac{7}{8}$ the Abelian Higgs model is not algebraically integrable (Yoshida 1983).

Toda–Brumer criterion. Since the Abelian Higgs model is not integrable for $\lambda^2<\frac{7}{8}$ it may be worthwhile to enquire if it exhibits chaos and, if so, what is the critical energy for the onset of chaos. This can be done by applying the Toda–Brumer criterion (Toda 1974, Brumer and Duff 1976) which essentially determines if the trajectories of two nearby points diverge exponentially in time. It turns out that if any eigenvalue λ of the 2×2 matrix $\partial^2U/\partial q_i\,\partial q_j$ (where $U(q_1, q_2)$ is the potential) is negative then the system shows exponential instability. It is easily seen that λ is negative provided

$$\gamma t - s^2 = 0 \tag{13}$$

where

$$\gamma \equiv \frac{\partial^2 U}{\partial q_1^2} \qquad t = \frac{\partial^2 U}{\partial q_2^2} \qquad s = \frac{\partial^2 U}{\partial q_1\,\partial q_2}. \tag{14}$$

In our case the potential $U(q_1, q_2)$ as given by (6) is

$$U(q_1, q_2) = \tfrac{1}{2}e^2q_1^2q_2^2 + c_4(q_2^2 - c_2/2c_4)^2. \tag{15}$$

Hence the Abelian Higgs model shows exponential instability if

$$\gamma t - s^2 = 12c_4q_2^4 - 2c_2q_2^2 - 3e^2q_1^2q_2^2 < 0. \tag{16}$$

This gives us

$$e^2q_{1,\min}^2 = 4c_4q_2^2 - \tfrac{2}{3}c_2 \tag{17}$$

beyond which the system shows instability. On substituting $q_{1,\min}^2$ in $U(q_1, q_2)$ given by (15) and minimising with respect to q_2 we obtain the critical energy

$$E_c = \tfrac{11}{108}c_2^2/c_4 \tag{18}$$

beyond which the Abelian model may exhibit chaos. It may be noted here that the Toda–Brumer criterion gives only a lower bound to the critical (threshold) energy for the onset of chaos.

Let us now consider the question of chaos in the SO(3) Georgi–Glashow model which has a monopole solution. The Lagrangian density for this model is given by

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}^aF^{a\mu\nu} + \tfrac{1}{2}D_\mu\phi^aD^\mu\phi^a - \tfrac{1}{4}\lambda(\phi^a\phi^a - m^2/\lambda)^2 \tag{19}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu \qquad a, b, c = 1, 2, 3 \tag{20}$$

$$D_\mu \phi^a = \partial_\mu \phi^a + g\varepsilon^{abc} A^b_\mu \phi^c. \tag{21}$$

The field equations are

$$D^\mu F^a_{\mu\nu} = -g\varepsilon^{abc} \phi^b D_\nu \phi^c \tag{22}$$

$$D^\mu D_\mu \phi^a = -\lambda \phi^a (\phi^b \phi^b - m^2/\lambda). \tag{23}$$

As in the Abelian Higgs model we work in the regime in which the fields are homogenous in space. In particular our ansatz is

$$A^a_0(\boldsymbol{x}, t) = 0 \qquad A^a_i(\boldsymbol{x}, t) = \varepsilon_{aij} a^j(t) \tag{24a}$$

$$\phi^a(\boldsymbol{x}, t) = \sqrt{2}\eta^a(t) \tag{24b}$$

$$a^1(t) = a^2(t) = a^3(t) = q_1(t) \tag{24c}$$

$$\eta^1(t) = \eta^2(t) = \eta^3(t) = q_2(t). \tag{24d}$$

The ansatz automatically takes care of the Gauss law constraint. The remaining field equations reduce to

$$\ddot{q}_1(t) = -6g^2 q_1 q_2^2 - 3g^2 q_1^3 \tag{25a}$$

$$\ddot{q}_2(t) = -6g^2 q_2 q_1^2 - 6\lambda q_2^3 + m^2 q_2. \tag{25b}$$

The corresponding Hamiltonian is

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + 3g^2 q_1^2 q_2^2 + \tfrac{3}{4} g^2 q_1^4 + \tfrac{3}{2}\lambda (q_2^2 - m^2/6\lambda)^2. \tag{26}$$

Proceeding as in the Abelian Higgs model one can perform the Painlevé test and calculate the resonances. It turns out that the leading-order exponents are again as given by (9a) with c_1 and c_2 now being given by

$$g^2 c_1^2 = \frac{2(g^2 - \lambda)}{3(\lambda - 2g^2)} \qquad c_2^2 = \frac{1}{3(\lambda - 2g^2)}. \tag{27}$$

In this case the resonances turn out to be

$$-1, 4, \tfrac{3}{2} \pm \sqrt{(2g^2 + 7)/(2g^2 - \lambda)}. \tag{28}$$

Two of the resonances are complex if $\lambda > 2g^2$ and hence the system is algebraically non-integrable in that region.

For the non-integrable region, we have again applied the Toda–Brumer criterion. We find that the SO(3) Georgi–Glashow model shows exponential instability if

$$18g^2 q_1^4 - 3m^2 q_1^2 + (54\lambda - 36g^2) q_1^2 q_2^2 - 2m^2 q_2^2 + 36\lambda q_2^4 < 0. \tag{29}$$

This is a complicated constraint and we are unable to solve it exactly and find E_c for the onset of chaos. We have therefore tried to calculate E_c for various values of q_1 and q_2 and among them we find that $q_1 = 0$ gives the lowest E_c. On putting $q_1 = 0$ in (29) we find that $q^2_{2,c} = m^2/18\lambda$. Using these values of q_1 and q_2 in the potential energy $U(q_1, q_2)$ given by (26) we find that the critical energy is $E_c = m^4/54\lambda$ beyond which the SO(3) Georgi–Glashow model exhibits chaos.

Thus we have seen that both the Abelian Higgs model and SO(3) Georgi–Glashow model may exhibit chaos for a range of parameter values. Before one can draw any firm conclusion one must perform a surface of section technique, calculate the Lyapunov exponents and study the stability of periodic solutions. We believe that these studies will confirm the chaotic behaviour of the systems (Kumar and Khare 1989) because *à la* Yang–Mills theory both Hamiltonians (see (6) and (26)) have the non-central potential $q_1^2q_2^2$. Because of this we also believe that both the Abelian Higgs model and SO(3) Georgi–Glashow model will also exhibit quantum chaos (Haller *et al* 1984, Pullen and Edmonds 1981) for a range of parameter values.

It is gratifying to note that in the Abelian Higgs model the chaos and vortices exist in two different regions, i.e. whereas vortices have been seen in type-II superconductors we find that chaos may exist in almost the entire type-I region. We suggest that one could look for the signature of chaos in the type-I region at large ρ ($\rho=\sqrt{x^2+y^2}$) where both gauge and Higgs fields are approximately uniform in space.

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